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Radiation damping of a quantum harmonic oscillator†

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Abstract. Using a fully coupled oscillator model for the quantum mechanical description of a harmonically bound electron in a radiation field, exact results about the dissipative properties of the system in thermal equilibrium are derived. In particular one gets, without any additional assumption, the classical equation of motion of an oscillating electron damped by radiation and the scattering cross section of the system for incident light.

1. Introduction

A central harmonic oscillator linearly coupled to a non-self-interacting system of a large number of harmonic field oscillators yields an often considered model for the study of dissipative effects. The common interest in it results from its universal character. Its applications are, for example, Brownian motion, interaction of matter with a field (of quanta, like photons, phonons) and quantum noise. It gives an approximate description of physical systems such as the laser and the polaron. From a mathematical point of view the model has the great advantage of permitting an exact treatment, see [1] and references therein. However, most applications need the limit of an infinite number of field oscillators to be considered. The problem of renormalisation then arises. Depending on the high-frequency behaviour of the spectral strength function, renormalisation may become inevitable in order to achieve finite expressions for observable quantities. This subtle point requires additional consideration, cf [2, 3].

Recently the model has been used for the study of a harmonically bound electron in a radiation field [2-5]. For this system the model arises from minimal coupling of the electron to the free quantised electromagnetic field

$$H = \frac{1}{2m}[\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{x})]^2 + \frac{m}{2}\eta^2\boldsymbol{x}^2 + \tfrac{1}{2}\int [\boldsymbol{E}(\boldsymbol{y})^2 + \boldsymbol{B}(\boldsymbol{y})^2]\,\mathrm{d}^3y$$

by (i) retaining only the dipole interaction and (ii) neglecting the self-interaction term of the field. Hence the electron-field interaction is described by the momentum-(vector) potential coupling $-(e/m)\boldsymbol{pA}$ at the centre of motion of the oscillating electron. One recognises the simplified system as the special case of the so-called fully coupled oscillator where the central oscillator couples uniformly to the field oscillators, see (7).

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In [2-5] this specified model has been treated exactly and explicit expressions for the partition function, the potential, kinetic and interaction energy in thermal equilibrium have been computed. Moreover, the state density of the system has been studied [3] showing the existence of a stable ground state and a line broadening of the excited states due to radiation.

In the present paper we pursue these investigations. We are interested in the following questions. How does a harmonically bound electron behave, being in thermal equilibrium with the ever-present radiation field, if it is subjected to an external force? How does it return to equilibrium after a perturbation? Which is the dissipation of work done on the oscillating electron by a driving force?

The following answers result from exact computations concerning the above model. A charged quantum oscillator, which is in thermal equilibrium with the surrounding radiation field, macroscopically behaves like a mechanical oscillator subjected to a mechanical friction. Left to itself it satisfies the equation of motion

$$\ddot{q} + \gamma\dot{q} + \eta^2 q = 0 \tag{1}$$

where $q = \langle p(t) \rangle$ or $q = \langle x(t) \rangle$ and where η is the natural frequency of the oscillator. This holds true even in the case of zero temperature where the oscillator couples to the vacuum modes. The damping coefficient we derive

$$\gamma = 2e^2\eta^2/(3mc^3) \tag{2}$$

is a constant, independent of time and temperature. The expression (2) agrees with electrodynamical considerations where γ describes the damping due to the emission of radiation by a classical charged oscillator, see e.g. [6, (41.6)]. (It is also in accord with QED perturbation theory (cf also [3]) where γ is the natural linewidth of the first excited state of a quantum oscillator.)

A further characteristic quantity of the system we derive concerns the scattering cross section σ_s for radiation. In thermal equilibrium the energy absorbed from an incident beam is re-emitted in all directions. For this scattering (=dissipative) process we get

$$\sigma_s(\omega) = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2} \tag{3}$$

where ω is the frequency of the incident radiation. This is well known from electrodynamics, see e.g. [6, (41.8)].

The formulae (1)-(3), which typically refer to the phenomenon of energy dissipation, are well known from classical physics, but no one has yet completely succeeded in giving a purely quantum mechanical explanation of them. One group of attempts starts with the classical equation of motion (1) and builds it into the Hamilton formalism in order to achieve a quantised version [7]. The difficulties which arise are either a time dependent oscillator mass or a violation of the uncertainty relations. Related to this approach is the introduction of non-linear terms into the Schrödinger equation which produce the right equation of motion for the expectation values [8], but it suffers from unstable stationary solutions. A third approach makes use of a quantum Langevin equation where an outer stochastic force simulates the microscopic interactions with a system of a large number of degrees of freedom [9]. Moreover there are many other attempts which we do not mention here. For an exhaustive review we refer to [10] and the literature cited therein.

A stochastic modelling of thermal and quantal fluctuations (as, e.g., mentioned above) provides a phenomenological description of damping. In [11] it is shown that a consistent stochastic modelling based on phenomenological considerations is possible. The basic assumptions in [11] are: (i) the expectation value of the position $\langle q(t)\rangle$ satisfies the classical irreversible equation of motion (1) and (ii) its response function coincides with the generalised susceptibility for the underlying quantum processes. The damping constant γ has the meaning of an effective (phenomenological) parameter which remains unspecified. Following the formalism, the Matsubara suceptibility (which is the central entity in this context) is derived from the generalised susceptibility due to a function theoretical one-to-one correspondence. Thus all autocorrelation functions are at hand. (For a comparison between some corresponding formulae presented here and in [11] one has to interchange position and momentum variables and substitute m for $(M\omega_0^2)^{-1}$, since in our model, resulting from minimal coupling, not the position but the momentum couples to the environment.)

We are interested in a microscopic understanding of the basic assumptions (i) and (ii) of the above phenomenological approach. A model which is well suited for this purpose is the oscillator-heat-bath model already mentioned. It has been studied by Ullersma in a series of papers and, more recently, has been thoroughly revisited in [1]. Our Hamiltonian H, equation (7), treated in § 2 follows also from a special case of that model. It is closely related to the case characterised by Ullersma's spectral function (see below), which has often been considered in the literature, e.g. [1, 12, 13]. Let us discuss this relation in some detail and then indicate the point to which we intend to make our specific contribution. For infinitely many field oscillators the spectral strength of (7) becomes $\gamma(\omega) = (2/\pi)\gamma\omega^2 l_c(\omega)$, where γ is a constant given by (2) and $l_c(\omega)$ is equal to 1 for ω not greater than the cutoff frequency ω_c and zero elsewhere. The quadratic growth is a consequence of the physical fact that the electric charge couples uniformly to the modes of the electromagnetic field. The cutoff is due to a Debye regularisation and has to be put to infinity afterwards. For H to have a finite lower bound the positivity condition must be satisfied which is $\eta^2 > (2/\pi)\gamma\omega_c$. This requires a finite cutoff. Only in the weak coupling limit, where the fine structure constant $e^2(\hbar c)^{-1}$ or, equivalently, γ tends to zero, is the cutoff allowed to tend to infinity. The corresponding limiting case for Ullersma's spectral strength $(2/\pi)\kappa\alpha^2\omega^2(\alpha^2+\omega^2)^{-1}$, i.e. the case where κ tends to zero and α obeys $\eta^2 > \alpha\kappa$, indicates that the central oscillator approaches the motion of damped oscillations with the damping coefficient γ, see § 7 of [1]. However, even if the electromagnetic coupling in some cases may be regarded as weak compared with the binding force of the central oscillator, the crucial problem is not touched thereby. For any non-zero coupling strength the theory depends on an almost arbitrary parameter, namely ω_c (corresponding to α in Ullersma's spectral strength). It cannot be put equal to infinity because of the occurrence of infinite energies and divergencies related to them, which are traced back to the already mentioned uniformity of the electromagnetic coupling. In order to get unambiguous results one has to get rid of this parameter. The problem is well known in QED and it is solved by the concept of renormalisation. The latter is inevitable in order to get observable finite differences in energies and, even in the case of convergence, to take account of the *observable* (not the bare) physical parameters. The renormalisation of the mass we shall perform is well known, e.g. from the theory of the natural linewidth. In the present case, besides the mass, the zero of the energy has also to be redefined. The latter causes a shift of the ground state from infinity to a finite value, see [2, 3] and § 4. As a consequence the kinetic and the potential energy

or, equivalently, the dispersions of the momentum and the position become finite. Let us mention that this is the context in which the divergency of the position dispersion in the stochastic model [11] can be physically understood. After renormalisation the limit $\omega_c \to \infty$ is performed. The results thus derived are valid without restrictions on the coupling strength and the temperature.

Closely related to our approach are [12, 13]. In [12] a fully coupled oscillator (or an even more general) potential with coupling constants c_α and frequencies ω_α of the field oscillators is discussed. In order to get a well defined friction coefficient $m\gamma$ it is assumed that $\Sigma_\alpha (m_\alpha \omega_\alpha^2)^{-1} c_\alpha^2 \omega^{-1} \delta(\omega - \omega_\alpha)$ is constant, equal to $(2/\pi)m\gamma$. Indeed, this assumption leads to (1). But it is by no means evident that it can be satisfied. In fact a solution arises from the model considered here in the limit of infinitely many field oscillators after having performed the renormalisations indicated above. In [13] a renormalisation of the mass and the frequency are considered but not carried through. So the dynamical equation of motion (1) of the damped oscillator does not strictly follow but only approximately with its validity restricted to small damping and certain frequency regimes.

In the following we get a clear-cut derivation of dissipation for an exactly solvable quantum mechanical model. The well known model is simple but nevertheless physically reasonable. The main tool used to obtain the results is a twofold renormalisation adopted from well known procedures in QED. Without any further assumptions it follows among others that the classical irreversible equation of motion (1) of a linearly damped oscillator emerges from a fully microscopic model.

The method we apply is functional integration. By averaging out the variables of the field oscillators one gets an effective action S_{eff} depending on a cutoff frequency ω_c. To get rid of the latter, two renormalisations have to be performed: (i) a mass renormalisation and (ii) a redefinition of the zero mark of the potential energy. Then the remaining path integral can be done yielding the reduced partition function $Z_F(\beta)$ at the inverse temperature β of the central oscillator. It is a functional of the outer generalised force $\boldsymbol{F}$ which couples linearly to the momentum or the position of the oscillator, giving rise to the additional term $\boldsymbol{YF}$, where $\boldsymbol{Y} = \boldsymbol{p}$ and $\boldsymbol{Y} = \boldsymbol{x}$ respectively, to the Hamilton operator. Because of the quadratic form of the action one has

$$Z_F(\beta) = Z_\alpha(\beta) \exp\left(\frac{1}{2\hbar} \int_0^\beta \int_0^\beta \sum_{j,k=1}^{3} M_{jk}(|t-s|) F_j(t) F_k(s)\, \mathrm{d}t\, \mathrm{d}s\right) \tag{4}$$

where $Z_\alpha(\beta)$, i.e. the partition function for $\boldsymbol{F} = 0$, has been computed in [2, 3]. From the Matsubara susceptibility

$$M_{jk}(\tau) = (\hbar \operatorname{Tr} U_\beta)^{-1} \operatorname{Tr}(Y_j U_\tau Y_k U_{\beta-\tau})$$

where

$$U_\tau := \exp(-\tau H/\hbar) \qquad 0 \leqslant \tau \leqslant \beta$$

the generalised susceptibility

$$\alpha_{jk}(t) := -2 \operatorname{Im} M_{jk}(it)\theta(t) = (\mathrm{i}/\hbar)\langle [Y_j(t), Y_k(0)] \rangle_\beta \theta(t)$$

follows by analytic continuation. By the fluctuation–dissipation theorem and linear response theory one gets the crucial relation

$$\langle Y_j(t) \rangle = \int_{-\infty}^{t} \sum_{k=1}^{3} \alpha_{jk}(t-s) F_k(s)\, \mathrm{d}s \tag{5}$$

which yields the macroscopic behaviour under the influence of the external force. As a consequence the time average of the work done per unit time on the central oscillator by a monochromatic force $\boldsymbol{F}_0 \cos \omega t$ is given by

$$Q = \frac{\mathrm{i}\omega}{4} \sum_{j,k=1}^{3} (\check{\alpha}^*_{jk}(\omega) - \check{\alpha}_{kj}(\omega)) F_{0j} F^*_{0k} \tag{6}$$

with

$$\check{\alpha}_{jk}(\omega) := \int_0^\infty \alpha_{jk}(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t.$$

2. Reduced partition function

The Hamiltonian under consideration is

$$H = \frac{1}{2m}\boldsymbol{p}^2 + \frac{m}{2}\eta^2 \boldsymbol{x}^2 + \sum_{k,\sigma}\left(\frac{1}{2\mu} p^2_{k\sigma} + \frac{\mu}{2}\omega_k^2 x^2_{k\sigma}\right) - \boldsymbol{p}\sum_{k,\sigma}\frac{e}{m}\left(\frac{4\pi\mu}{\Omega}\right)^{1/2} \boldsymbol{u}_{k\sigma} x_{k\sigma}. \tag{7}$$

Here m, η and e denote the mass, frequency and charge of the central oscillator $\boldsymbol{x}$, $\boldsymbol{p}$. It couples to the one-dimensional field oscillators $x_{k\sigma}$, $p_{k\sigma}$ with frequencies ω_k. The orthonormal polarisation vectors $\boldsymbol{u}_{k\sigma}$, $\sigma = 1, 2$, are perpendicular to the lattice vectors $\boldsymbol{k}$ corresponding to the finite quantisation volume Ω. μ is a constant introduced for dimensional reasons only. H arises from minimal coupling of a harmonically bound electron to free radiation field [2-5]. Following [2, 3], the effective action for the central oscillator is found to be

$$S_{\mathrm{eff}}[\boldsymbol{q}, \boldsymbol{F}] = \int_0^\beta \left(\frac{1}{2m\eta^2}\dot{\boldsymbol{q}}^2 + \frac{1}{2m}\boldsymbol{q}^2 + \boldsymbol{q}\boldsymbol{F}\right)\mathrm{d}t$$
$$+ \frac{-e^2}{3\pi m^2 c^3}\int_0^\beta \int_0^\beta \left(\int_0^{\omega_c} \frac{\omega \cosh[\omega(|t-s|-\beta/2)]}{\sinh(\omega\beta/2)}\,\mathrm{d}\omega\right)\boldsymbol{q}(t)\boldsymbol{q}(s)\,\mathrm{d}t\,\mathrm{d}s \tag{8}$$

averaging out the field variables. The cutoff frequency ω_c comes in replacing $\Sigma_{k,\sigma}$ with $\Omega(2\pi c)^{-3}\Sigma_\sigma \int \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \int_0^{\omega_c} \omega^2\,\mathrm{d}\omega$. To get rid of it a renormalisation has to be carried out. It is convenient to perform the mass renormalisation already now. It amounts to the substitution

$$[1 - 4e^2\omega_c(3\pi m c^3)^{-1}]m^{-1} \to m^{-1} \tag{9}$$

cf [2, 3] for details. We point out that the generalised force $\boldsymbol{F}$ in (8) couples to the momentum of the electron. According to (4) one gets for the reduced partition function

$$Z_F(\beta) = Z_\alpha(\beta)\exp\left(\frac{1}{2\hbar}\int_0^\beta\int_0^\beta M^{\mathrm{mom}}(|t-s|)\boldsymbol{F}(t)\boldsymbol{F}(s)\,\mathrm{d}t\,\mathrm{d}s\right) \tag{10}$$

where the Matsubara susceptibility is a scalar since H is isotropic. Explicitly one gets, essentially by minimising (8), cf [5],

$$M^{\mathrm{mom}}(\tau) = \frac{m}{\beta}\sum_{n=-\infty}^{\infty}(K_n + \nu_n^2/\eta^2)^{-1}\exp(-\mathrm{i}\nu_n\cdot\tau) \tag{11}$$

with

$$K_n := 1 + 4e^2\nu_n(3\pi m c^3)^{-1}\tan^{-1}(\omega_c/\nu_n) \qquad K_0 := 1 \qquad \nu_n := 2\pi n/\beta.$$

Performing the limit $\omega_c \to \infty$ and summing up the resulting series [14, (14.3.1), (17.3.1)] one ends up with

$$M^{\mathrm{mom}}(\tau) = \frac{m\rho}{\beta \sin\varphi} \operatorname{Im}\Big([(\rho\, e^{i\varphi})^{-1} - 2\boldsymbol{\beta}(\rho\, e^{i\varphi})] \cos(\rho\, e^{i\varphi}(\pi - 2\pi\tau/\beta)) - \int_{2\pi\tau/\beta}^{\pi} \cot(x/2) \cos(\rho\, e^{i\varphi}(x - 2\pi\tau/\beta))\, dx \Big) \tag{12}$$

for $0 \leq \tau \leq \beta$, where

$$\boldsymbol{\beta}(x) := \sum_{k=0}^{\infty} (-1)^k (x+k)^{-1}$$

denotes the $\boldsymbol{\beta}$ function, and where we use the abbreviations

$$\rho := \beta\eta(2\pi)^{-1} \qquad \cos\varphi := e^2\eta(3mc^3)^{-1}. \tag{13}$$

The prefactor $Z_\alpha(\beta)$ has to be renormalised before performing the limit $\omega_c \to \infty$. To this the free energy or, equivalently, the internal or potential energy is shifted by

$$-\hbar e^2 \eta^2 (\pi m c^3)^{-1} \ln(1 + \omega_c/\eta). \tag{14}$$

The result for $Z_\alpha(\beta)$ is given in [2, 3], cf also [4, 5].

Because of the commutation relation $[H, p_j] = i\hbar m\eta^2 x_j$, which persists after the mass renormalisation, the Matsubara susceptibility for the position satisfies

$$M^{\mathrm{pos}} = (m^2\eta^4)^{-1} \ddot{M}^{\mathrm{mom}}. \tag{15}$$

3. Generalised susceptibility, autocorrelation function

As mentioned, from the Matsubara susceptibility the generalised susceptibility follows by analytic continuation, cf (4). The real part of $M(it)$ is equal to the symmetrised autocorrelation function

$$c(t) := (1/2\hbar)\langle (Y(t)Y(0) + Y(0)Y(t)) \rangle_\beta.$$

The latter will reveal the timescales of decay of the correlations which are due to damping and thermal relaxation.

Separating the real and imaginary part of $M^{\mathrm{mom}}(it)$ one gets

$$\alpha^{\mathrm{mom}}(t) = \frac{m\eta}{\sin\varphi} e^{-t\eta\cos\varphi} \sin(t\eta \sin\varphi)\theta(t) \tag{16}$$

$$c^{\mathrm{mom}}(t) = D^{\mathrm{mom}}(|t|) + T^{\mathrm{mom}}(|t|)$$

with

$$D^{\mathrm{mom}}(t) := \frac{m\eta}{2\sin\varphi} |\cot(\pi\rho\, e^{i\varphi})| \sin[t\eta \sin\varphi - \arg(\cot(\pi\rho\, e^{i\varphi}))]\, e^{-t\eta\cos\varphi} \tag{17}$$

$$T^{\mathrm{mom}}(t) := -\frac{2}{\pi} m\eta \cos\varphi \sum_{k=1}^{\infty} \frac{k\rho^2}{\rho^4 + k^4 - 2\rho^2 k^2 \cos 2\varphi} e^{-(2\pi/\beta)kt}.$$

From (15) it follows

$$\alpha^{\mathrm{pos}}(t) = \frac{1}{m\eta \sin\varphi} \mathrm{e}^{-t\eta\cos\varphi} \sin(2\varphi - t\eta \sin\varphi)\theta(t) \tag{18}$$

$$c^{\mathrm{pos}}(t) = D^{\mathrm{pos}}(|t|) + T^{\mathrm{pos}}(|t|)$$

with

$$D^{\mathrm{pos}}(t) := (2m\eta \sin\varphi)^{-1} |\cot(\pi\rho\, \mathrm{e}^{\mathrm{i}\varphi})| \sin[-t\eta \sin\varphi + 2\varphi + \arg(\cot(\pi\rho\, \mathrm{e}^{\mathrm{i}\varphi}))]\, \mathrm{e}^{-\eta t\cos\varphi} \tag{19}$$

$$T^{\mathrm{pos}}(t) := \frac{2\cos\varphi}{\pi m\eta} \sum_{k=1}^{\infty} \frac{k^3}{\rho^4 + k^4 - 2\rho^2 k^2 \cos 2\varphi} \mathrm{e}^{-(2\pi/\beta)kt}.$$

First of all, one realises that the generalised susceptibilities are independent of the temperature (!) and satisfy the differential equation (1) with γ equal to (2). By (5), the temporal behaviour of the expectation values $\boldsymbol{q}(t) = \langle \boldsymbol{p}(t) \rangle_\beta$ and $\boldsymbol{q}(t) := \langle \boldsymbol{x}(t) \rangle_\beta$ is given by

$$\int_{-\infty}^{t} \alpha^{\mathrm{mom/pos}}(t-s)\boldsymbol{F}(s)\, \mathrm{d}s$$

where $\boldsymbol{F}$ is a generalised external force coupling to the momentum and position respectively. In particular this means that after a temporally limited perturbation the system relaxes to equilibrium by a damped harmonic oscillation according to (1).

The Fourier inverses of the generalised susceptibilities are

$$\check{\alpha}^{\mathrm{mom}}(\omega) = \frac{m\eta^2(\eta^2 - \omega^2)}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2} + \mathrm{i}m\eta^2\gamma \frac{\omega}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2} \tag{20}$$

$$\check{\alpha}^{\mathrm{pos}}(\omega) = \frac{1}{m\eta^2} \frac{\eta^2(\eta^2 - \omega^2) + \gamma^2\omega^2}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2} + \mathrm{i}\frac{\gamma}{m\eta^2} \frac{\omega^3}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2}. \tag{21}$$

According to (6), the imaginary parts of (20) and (21) determine the energy dissipation. In the case of an incident electromagnetic plane wave the interaction is approximately described by $\boldsymbol{pA}$ or $\boldsymbol{xE}$ coupling at the centre of motion of the oscillating electron. The incident energy flux for the plane wave is the time-averaged Poynting vector, namely $c|E_o|^2(8\pi)^{-1}$. So the cross sections are

$$\sigma^{\mathrm{mom}}(\omega) = \sigma_0 \frac{\eta^4}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2} \qquad \text{for the } \boldsymbol{pA} \text{ coupling} \tag{22}$$

$$\sigma^{\mathrm{pos}}(\omega) = \sigma_0 \frac{\omega^4}{(\omega^2 - \eta^2)^2 + \gamma^2\omega^2} \qquad \text{for the } \boldsymbol{xE} \text{ coupling} \tag{23}$$

where $\sigma_0 := (8\pi/3)e^4(mc^2)^{-2}$ is the Thomson cross section for a free electron. It is σ^{pos} that coincides with σ_s, equation (3). Around $\omega = \eta$, both (22) and (23) are very well approximated by $\sigma_0\eta^2[4(\omega - \eta)^2 + \gamma^2]^{-1}$. Differences between (22) and (23) arise for small frequencies, since $\sigma^{\mathrm{mom}}(0) = \sigma_0$ and $\sigma^{\mathrm{pos}}(0) = 0$, and for high frequencies, since $\sigma^{\mathrm{mom}}(\infty) = 0$ and $\sigma^{\mathrm{pos}}(\infty) = \sigma_0$.

Concerning the symmetrised autocorrelation functions one recognises D as the damping term and T as the thermal decay term. In D only the phase and the amplitude depend on the temperature. The decay time is $2/\gamma$, which is independent of the temperature. The difference between D^{mom} and D^{pos} essentially consists in an additional phase.

For the momentum correlation the thermal decay time at finite temperatures is $\beta/(2\pi)$. At zero temperature (17) shows a different behaviour. One has

$$c^{\mathrm{mom}}(t, \beta^{-1}=0)=\frac{m\eta}{2\sin\varphi}\sin\left(|t|\eta\sin\varphi+\frac{\pi}{2}\right)\mathrm{e}^{-|t|\eta\cos\varphi}$$
$$-\frac{2}{\pi}m\eta\cos\varphi\int_0^\infty\frac{x\exp(-|t|\eta x)}{x^4-2x^2\cos 2\varphi+1}\,\mathrm{d}x. \tag{24}$$

In this case the asymptotic expansion of the thermal decay term reveals a merely algebraic decay:

$$-(2/\pi)m\eta\cos\varphi[(\eta t)^{-2}+(12\cos 2\varphi)(\eta t)^{-4}+\dots].$$

At high temperatures c^{mom} behaves like

$$m\eta(2\pi\sin\varphi)^{-1}\rho^{-1}\sin(\varphi+|t|\eta\sin\varphi)\exp(-|t|\eta\cos\varphi)$$

i.e. the damped oscillation exceeds the thermal decay. Finally,

$$3\hbar(2m)^{-1}c^{\mathrm{mom}}(0)=(3/2)\hbar\eta[(\pi\sin\varphi)^{-1}\operatorname{Im}\psi(\rho c^{\mathrm{i}\varphi})-(2\pi\rho)^{-1}]$$

which is the kinetic energy at thermal equilibrium, cf [4, 5].

For the position correlation the thermal decay time at finite temperatures is also $\beta/(2\pi)$. At zero temperature (19) becomes

$$c^{\mathrm{pos}}(t, \beta^{-1}=0)=(2m\eta\sin\varphi)^{-1}\sin(2\varphi-\pi/2-|t|\eta\sin\varphi)\mathrm{e}^{-|t|\eta\cos\varphi}$$
$$+\frac{2\cos\varphi}{\pi m\eta}\int_0^\infty\frac{x^3\exp(-|t|\eta x)}{x^4-2x^2\cos 2\varphi+1}\,\mathrm{d}x. \tag{25}$$

Again the thermal decay is merely algebraic:

$$2(\pi m\eta)^{-1}\cos\varphi[6(\eta t)^{-4}+(240\cos 2\varphi)(\eta t)^{-6}+\dots].$$

Like c^{mom}, at high temperatures c^{pos} essentially undergoes a damped harmonic oscillation

$$(2\pi m\eta\sin\varphi)^{-1}\rho^{-1}\sin(\varphi-|t|\eta\sin\varphi)\exp(-|t|\eta\cos\varphi).$$

However, at zero time c^{pos} becomes infinite. We will discuss this point.

4. Renormalisation of the potential energy

As indicated in (13), we had to shift the potential energy in order to get a finite partition function $Z_\alpha(\beta)$. However, the renormalisation of the potential energy is not yet accomplished by this. This follows from the fact that, as a function of the cutoff ω_c, the Matsubara susceptibility M^{pos} at $\tau=0$ diverges like (14) as ω_c tends to infinity (cf [5]), instead of yielding the potential energy (up to the factor $\frac{3}{2}\hbar m\eta^2$). So it is not surprising that M^{pos}, equation (15), has a singularity at $\tau=0$. This singularity is the reason for $c^{\mathrm{pos}}(0)$ not being finite. It is also responsible for another inconsistency we have not yet mentioned, namely the fact that α^{pos} does not vanish for $t\searrow 0$ as it should according to its definition (4).

Thus we are led to renormalise the potential energy removing the singularity of M^{pos} at $\tau = 0$, which is

$$-2(\pi m\eta)^{-1} \cos\varphi[C + \ln(\tau\eta)]. \tag{26}$$

C is Euler's constant. The choice of the constant term is consistent with (14).

The modifications arising from this renormalisation are, first of all, that $c^{\text{pos}}(0)$ becomes finite, namely

$$\tfrac{3}{2}\hbar m\eta^2 c^{\text{pos}}(0) = \tfrac{3}{2}\hbar\eta[-(\pi \sin\varphi)^{-1} \operatorname{Im}(e^{2i\varphi}\psi(\rho\, e^{i\varphi})) + (2/\pi)\cos\varphi \ln\rho - (2\pi\rho)^{-1}]$$

which is the potential energy, cf [4, 5].

Then, $\frac{3}{2}\hbar m\eta^2 c^{\text{pos}}$ no longer vanishes for $t \to \infty$ since the renormalisation term

$$(3/\pi)\hbar\eta \cos\varphi(C + \ln|t|\eta)$$

increases. However, the latter stays small for a long time. Indeed, for the electron mass and the ultraviolet value $\eta = 10^{16}\,\text{s}^{-1}$, after one year it amounts to about one-millionth of the ground state energy $\frac{3}{2}\hbar\eta$.

Finally, α^{pos} becomes continuous at $t = 0$ by the subtraction of the constant $2(m\eta)^{-1}\cos\varphi$ for $t > 0$. The effect is only a constant shift of the centre of oscillation.

Of course, $[H, p_j] = i\hbar m\eta^2 x_j$ is no longer valid.

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